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Some applications of lie transformation groups to classical Hamiltonian dynamics

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SOME APPLICATIONS OF LIE TRANSFORMATION GROUPS TO CLASSICAL HAMILTONIAN DYNAMICS

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CHAPTER I

INTRODUCTION

1. Summary of Results

The last twenty or thirty years have seen a great increase in the use of group theoretical techniques in mathematical physics. One important reason for this is that group theory is a natural vehicle for the analysis of the physical and mathematical symmetries of classical and quantum mechanical problems[1]. The knowledge of symmetries of a physical system can lead to non-trivial results, as evidenced by Noether's theorem[2].

A fundamental role in these symmetry analyses is played by the pioneering work of S. Lie[3]. Lie developed analytical techniques for determining groups of transformations under which a differential equation is covariant (invariant in form); since most physical problems are formulated in the language of differential equations, Lie's techniques provide a tool eminently suited for the discussion of the symmetries of a physical problem.

Recent work[4] has established that a group theoretical viewpoint of completely integrable dynamical systems with N degrees of freedom yields an algorithm that provides new information concerning the symmetry transformation group structure of this class of dynamical systems. The work presented here rests heavily on the results presented in reference [4] and it is recommended that the reader consult this reference for a more rigorous discussion of the results given in this thesis.

This thesis presents the following results:

1. If one knows the solution to the initial value problem for two completely integrable Hamiltonian dynamical systems with the same number of degrees of freedom and with a common Cartan state space, then one can write down a transformation which globally connects these two systems. (Chapter II)

2. A transformation between two Hamiltonian dynamical systems with the same number of degrees of freedom transforms Lie symmetry generators of one system into Lie symmetry generators of the other system; in addition, the Lie algebraic structure of these generators is preserved under this transformation. (Chapter III)

3. One may always find a Lie generator associated with a given constant of the motion of a Hamiltonian dynamical system; a technique is presented whereby one may be able to find a conserved quantity associated with a given Lie generator. (Chapter V)

4. A transformation group isomorphic to the group SL(N+2,R) is found to be a symmetry transformation group of the classical isotropic harmonic oscillator problem with N degrees of freedom. The group transformations are given explicitly. (Chapter VI)

It is hoped that this work will give further evidence as to the power of the group theoretical viewpoint and will help stimulate others to use these methods and to develop new ones to use in the study of differential equations.

2. Notation

Throughout this work we use the Einstein summation convention; unless otherwise indicated, indices are understood to take on integer values from 1 to N.

Hamilton's equations of motion for a system with N degrees of freedom described by the Hamiltonian function H(x,p,t) are

$$\begin{aligned} \dot{x}_i &= \{x_i, H\} = H_{p_i} \\ \dot{p}_i &= \{p_i, H\} = -H_{x_i} \end{aligned} \tag{1.1}$$

where the bracket is the Poisson bracket, an overdot denotes total differentiation with respect to t, and a variable written as a subscript denotes partial differentiation with respect to that variable. We also use the vector notation $x = (x_1, \ldots, x_N)$ and $p = (p_1, \ldots, p_N)$. It is important to note that here the x_i are Euclidean space coordinates with the canonically conjugate momenta p_i; this is emphasized by using x rather than the generalized coordinate set q usually associated with Hamiltonian dynamics.

We use the operator $\{\cdot,F\}$ which we call the Poisson bracket operator:

$$\{\cdot\,,F\} = F_{p_i}\partial_{x_i} - F_{x_i}\partial_{p_i}$$

such that $\{\cdot,F\}G = \{G,F\}$ where F and G are arbitrary functions of x, p, and t. If F is the Hamiltonian function H of a physical system, then we call $\{\cdot,H\}$ the Liouville operator[5].

The Lie-type infinitesimal transformations discussed in this work are written

$$\begin{aligned} t' &= t + \varepsilon\,\xi(x,p,t) \\ x_i' &= x_i + \varepsilon\,\eta^{x_i}(x,p,t) \\ p_i' &= p_i + \varepsilon\,\eta^{p_i}(x,p,t) \end{aligned} \qquad (1.2)$$

where ε is some infinitesimal parameter. The Lie operator which represents this transformation is thus

$$U = \xi\partial_t + \eta^{x_i}\partial_{x_i} + \eta^{p_i}\partial_{p_i} \; . \qquad (1.3)$$

We also use the once-extended operator for our discussions:

$$U^{(1)} = U + (\dot{\eta}^{x_i} - H_{p_i}\dot{\xi})\partial_{\dot{x}_i} + (\dot{\eta}^{p_i} + H_{x_i}\dot{\xi})\partial_{\dot{p}_i} \; . \qquad (1.4)$$

A necessary and sufficient condition that the transformation (1.2) leave Hamilton's equations (1.1) invariant in form (i.e., covariant) is

$$\left.\left[U^{(1)}\left(\dot{x}_i - \{x_i, H\}\right)\right]\right|_{(1.1)} = 0$$
$$\left.\left[U^{(1)}\left(\dot{p}_i - \{p_i, H\}\right)\right]\right|_{(1.1)} = 0 \qquad (1.5)$$

where the subscripted (1.1) means that $\dot{x}_i$ and $\dot{p}_i$ are to be replaced by $\partial H/\partial p_i$ and $(-\partial H/\partial x_i)$, respectively, after the operation of $U^{(1)}$. Written out explicitly, these equations are

$$\dot{\eta}^{x_i} - H_{p_i}\dot{\xi} - H_{p_i t}\xi - H_{p_i x_j}\eta^{x_j} - H_{p_i p_j}\eta^{p_j} = 0 \qquad (1.6a)$$

$$\dot{\eta}^{p_i} + H_{x_i}\dot{\xi} + H_{x_i t}\xi + H_{x_i x_j}\eta^{x_j} + H_{x_i p_j}\eta^{p_j} = 0\,. \qquad (1.6b)$$

Ovsjannikov[1] has given the general solution to these equations.

CHAPTER II

RELATING TWO HAMILTONIAN SYSTEMS

1. The Equations of Motion as Lie Equations

If H(x,p) is the Hamiltonian function for a conservative mechanical system (i.e., H does not depend on t explicitly), the Hamiltonian equations of motion (1.1) can be written

$$\frac{dx_i}{H_{p_i}(x,p)} = \frac{dp_i}{-H_{x_i}(x,p)} = dt \quad (\text{no sum}) \qquad (2.1)$$

and we observe that these are nothing more than the equations used to obtain the finite transformations of a one parameter continuous transformation group whose infinitesimal generator is $\{\cdot,H\} = H_{p_i}\partial_{x_i} - H_{x_i}\partial_{p_i}$. These finite transformations can be written

$$x_i' = \Omega_i(x,p;t)$$

$$p_i' = \Gamma_i(x,p;t)$$

where t is the group parameter. Without loss of generality, one may choose t = 0 to correspond to the identity transformation; if we also denote the initial conditions to the mechanical problem at t = 0 by x_0 and p_0, we can then write

$$x_i = \Omega_i(x_o, \rho_o; t)$$
$$\rho_i = \Gamma_i(x_o, \rho_o; t) \qquad (2.2)$$

which is the solution to the problem. This approach emphasizes the role of the time as a group parameter rather than as an independent variable in an initial value problem. The group property of the solutions to this initial value problem is well-known; the reader interested in a rigorous discussion is referred to Hirsch and Smale[6].

We note a characteristic of this group property that is of great computational importance. It is not necessary to algebraically invert equations (2.2) in order to obtain the inverse transformation. Instead, we can use the group property to write

$$x_{io} = \Omega_i(x, \rho; -t)$$
$$\rho_{io} = \Gamma_i(x, \rho; -t).$$

This is a neat and elegant trick; see the acknowledgement to J. Turner in reference [4].

For example, we have the isotropic harmonic oscillator solutions (for unit frequency)

$$x_i = x_{i_0} \cos t + p_{i_0} \sin t$$

$$p_i = -x_{i_0} \sin t + p_{i_0} \cos t,$$

which by the above rule immediately yield the inverse transformation

$$x_{i_0} = x_i \cos t - p_i \sin t$$

$$p_{i_0} = x_i \sin t + p_i \cos t .$$

This is easier to interpret if one notices that these are the equations of rotation of each of the phase planes x_i, p_i.

Equations (2.2) represent a continuous transformation from arbitrary x(t), p(t) to the initial conditions x_0, p_0; we note that this is what is sought in the Hamilton-Jacobi formulation of mechanics. Now the central role played by the Liouville operator becomes apparent:

$$x_i = \exp[t\{\cdot, H\}]\, x_{i_0} ,$$

$$p_i = \exp[t\{\cdot, H\}]\, p_{i_0} \tag{2.3}$$

is a <u>symbolic</u> solution to the Hamilton-Jacobi problem, where we must remember that the exponential represents the corresponding power series in $t\{\cdot,H\}$[7] and only has meaning if H_{x_i} and H_{p_i} are analytic in x and p. (It is well-known that equations (2.3) represent a symbolic solution to Hamilton's equations of motion). Explicitly writing out equations (2.3), one obtains Taylor series expansions of $x_i(t)$ and $p_i(t)$; it is obvious that (2.3) are calculationally useless in all but the most simplest of physical problems (e.g., the free particle and the simple harmonic oscillator), at least if one desires the solution in closed form. Nevertheless, this is a very compact method for symbolically writing the solution to the Hamilton-Jacobi problem.

The group theoretical viewpoint provides a more conventional (and, possibly, more satisfying) theory of variable changes for the Hamiltonian equations of motion than does the standard generating function approach of the Hamilton-Jacobi theory. This is because the generating function mixes up the old and new canonical variables in implicit functions; the group theoretical viewpoint gives us one set of canonical variables directly as explicit functions of the other set of canonical variables.

As noted in the previous section, however, this group theoretical viewpoint is most definitely not new, but is ultimately due to Lie; for a modern interpretation, see for example Hirsch and Smale[6]. The <u>use</u> of this viewpoint to derive new results is presented here and in reference [4]. Reference [4] gives the theoretical foundation for the work presented here.

One might claim that there are many important non-conservative systems of (at least mathematical) interest that are not covered by the above theory. However, these cases can be investigated by the above techniques by introducing a new group parameter and letting t become a new canonical variable with the canonically conjugate momentum variable (-H) [8],[9]; thus every non-conservative Hamiltonian system with N degrees of freedom can be formally related to a mathematically conservative system with N + 1 degrees of freedom.

This parametric form of the Hamiltonian equations of motion in conjunction with Lie's theory of continuous transformation groups provides a truly elegant viewpoint of classical Hamiltonian dynamics.

2. Local One-to-One Maps Between Two Hamiltonian Systems

By the use of a change of variables, one may be able to locally transform a given Hamiltonian system into an equivalent (possibly Hamiltonian) system of the same number of degrees of freedom whose equations of motion have an explicitly different form (see Appendix I). The usual goal is to transform to a new Hamiltonian system whose equations of motion allow immediate integration. One may also study the converse problem: start with a "simple" Hamiltonian system whose solutions are known and transform to a more complicated system; however, this is obviously not a systematic method for attacking a given problem. Nevertheless, we will see that this approach does have theoretical importance in regards to the study of the group theoretical structure of classical Hamiltonian dynamical systems.

We can construct transformations such that one Hamiltonian system is transformed into another system, possibly also Hamiltonian in form. As an example, we exhibit a transformation between the free particle problem and the harmonic oscillator problem. The transformation

$$X = \rho\cos\left(\frac{-x}{\rho}\right) \qquad\qquad x = \sqrt{X^2+P^2}\,\tan^{-1}\frac{P}{X}$$
$$\text{OR} \qquad\qquad (2.4)$$
$$P = \rho\sin\left(\frac{-x}{\rho}\right) \qquad\qquad \rho = \sqrt{X^2+P^2}$$

takes the one degree of freedom free particle equations of motion $\dot{x} = p$, $\dot{p} = 0$ into the one degree of freedom harmonic oscillator equations of motion $\dot{X} = P$, $\dot{P} = -X$. Transformations such as these are not difficult to construct. We illustrate one method here that is reminiscent of the material in the previous section.

Suppose we have two Hamiltonian systems Σ_1 and Σ_2 which have the same number of degrees of freedom. To construct a transformation between these two systems, we first find a transformation for the system Σ_1 which will transform it to constant canonical variables. We then find another transformation which will transform system Σ_2 to constant canonical variables. We then put the two system's constant canonical variables in one-to-one correspondence and equate corresponding pairs. Since we now have two transformations which both involve common variables, we can eliminate the common variables to obtain the product transformation. Thus we will have a transformation from system Σ_1 to system Σ_2 that is locally one-to-one.

We easily construct a transformation of this type for transforming from the free particle problem with N degrees of freedom to the isotropic harmonic oscillator problem with N degrees of freedom. We do this by writing down the solutions to these problems and then interpreting the initial conditions for both problems as the constant canonical variables. We thus have the transformation

$$x_i = (\cos\omega t + t\sin\omega t)X_i - \omega^{-1}(\sin\omega t - t\cos\omega t)P_i$$

$$p_i = X_i \sin\omega t + \omega^{-1} P_i \cos\omega t \tag{2.5a}$$

which transforms the Hamiltonian equations of motion $\dot{x}_i = p_i$, $\dot{p}_i = 0$ into the Hamiltonian equations $\dot{X}_i = P_i$, $\dot{P}_i = -\omega^2 X_i$. The inverse of this transformation is

$$X_i = x_i \cos\omega t + (\sin\omega t - t\cos\omega t)p_i$$

$$P_i = -\omega x_i \sin\omega t + \omega(\cos\omega t + t\sin\omega t)p_i \quad . \tag{2.5b}$$

We note that the Jacobian of this transformation is identically equal to unity.

We call the transformations derived in this manner (i.e., using the solutions to the initial value problem) solution transformations. Thus, given any two systems with the same number of degrees of freedom whose solutions we know, we can theoretically write down a transformation connecting these two systems; the word theoretically is used since it is entirely conceivable that an initial value problem's solution could be too complicated for the application of the above method.

In a truly rigorous study, fundamental topological considerations would have to be taken into account; this is beyond the scope of the present work. For many physical problems these considerations are not crucial. One can easily imagine, however, systems where constraints would force an examination of these issues.

For a more detailed discussion of the ideas presented above, see reference [4]. Here, it is sufficient to note that what we are essentially doing is finding a common point in the common Cartan state space (the space of the variables x, p, t) of the two systems. Since one and only one trajectory for each system may pass through any point, we know that there is only one trajectory for each system through this common point. The group property of the solutions allows us to pass from any point on one of these trajectories through the common point to any other point on either of these trajectories (strictly, this is only true for completely integrable Hamiltonian systems; see reference [4]).

So far, we have not changed the parameter t in the transformations; there is no a priori reason why we should not. An important class of transformations of this type is given by a result due to G. Bluman[10].

Suppose we have an isotropic physical system which is described by N second order linear homogeneous ordinary differential equations. If the solutions to these differential equations can be written as $x^i(t) = c_1^i x_1(t) + c_2^i x_2(t)$, where $x_1(t)$ and $x_2(t)$ are two linearly independent solutions for <u>all</u> of the N differential equations and c_1^i and c_2^i are constants, then by dividing the equation for $x^i(t)$ by $x_2(t)$, we obtain $z^i = c_1^i\theta + c_2^i$, where $z^i = x^i(t)/x_2(t)$ and $\theta = x_1(t)/x_2(t)$. This then gives a transformation between this system and the N degree of freedom free particle problem $d^2z^i/d\theta^2 = 0$.

In particular, here we use the following transformation, which we will call the Bluman transformation, to connect the free particle system and the isotropic harmonic oscillator system with the same number of degrees of freedom:

$$
\begin{aligned}
t &= \tan \omega T \\
x_i &= X_i \sec \omega T \qquad\qquad (2.6)\\
p_i &= X_i \sin \omega T + \omega^{-1} P_i \cos \omega T
\end{aligned}
$$

which takes the free particle equations of motion $\dot{x}_i = p_i$, $\dot{p}_i = 0$ into the harmonic oscillator equations of motion

$$\frac{dX_i}{dT} = P_i \quad , \quad \frac{dP_i}{dT} = -\omega^2 X_i \quad .$$

The inverse of this transformation is

$$T = \omega^{-1} \tan^{-1} t$$

$$X_i = x_i A$$

$$P_i = -\omega x_i t A + \omega p_i A^{-1}$$

where $A = (1 + t^2)^{-1/2}$. We also note that the Jacobian of this transformation is

$$\frac{\partial(x, p, t)}{\partial(X, P, T)} = \frac{1}{\cos^2 \omega T}$$

so that the transformation is local in character.

The transformations considered in this section are, in general, local transformations. However, for completely integrable Hamiltonian dynamical systems, one has that the solution transformation is a global transformation. In combination with the results of the next chapter, we can then make general statements about the overall group theoretical structure of this class of dynamical systems. See reference [4] for more detailed comments concerning this structure.

CHAPTER III

TRANSFORMATION PROPERTIES OF THE LIE DETERMINING EQUATIONS

This chapter investigates the transformation properties of the Lie determining equations (1.5). Suppose we have a transformation

$$\begin{aligned} T &= \theta(x,p,t) \\ X_i &= \phi_i(x,p,t) \\ P_i &= \psi_i(x,p,t) \end{aligned} \quad , \quad \frac{\partial(X,P,T)}{\partial(x,p,t)} \neq 0 \qquad (3.1)$$

which transforms Hamilton's equations of motion (1.1) into the equations

$$\begin{aligned} f_i(X,P,T)\left(\frac{dX_i}{dT} - \{X_i,K\}\right) &= 0 \quad (\text{no sum}) \\ g_i(X,P,T)\left(\frac{dP_i}{dT} - \{P_i,K\}\right) &= 0 \quad (\text{no sum}) \end{aligned} \qquad (3.2)$$

where the Poisson brackets are calculated with respect to the variables X_i and P_i. Thus everywhere f_i and g_i are non-zero, the transformation (3.1) is a covariance transformation of Hamilton's equations (1.1) if the function K(X,P,T) exists.

Now if we use the transformation (3.1) to transform equations (1.5), we get

$$\left\{\tilde{U}^{(1)}\left[f_i(X,P,T)\left(\frac{dX_i}{dT} - \{X_i, K\}\right)\right]\right\}\Bigg|_{(3.2)} = 0 \text{ (no sum)}$$

$$\left\{\tilde{U}^{(1)}\left[g_i(X,P,T)\left(\frac{dP_i}{dT} - \{P_i, K\}\right)\right]\right\}\Bigg|_{(3.2)} = 0 \text{ (no sum)} \qquad (3.3)$$

where by the chain rule for partial differentiation the operator $\tilde{U}^{(1)}$ is

$$\tilde{U}^{(1)} = (U^{(1)}T)\big|_{X,P,T}\,\partial_T + (U^{(1)}X_i)\big|_{X,P,T}\,\partial_{X_i} + (U^{(1)}P_i)\,\partial_{P_i} + (U^{(1)}X_i')\big|_{X,P,T,X',P'}\,\partial_{X_i'} + (U^{(1)}P_i')\big|_{X,P,T,X',P'}\,\partial_{P_i'} \qquad (3.4)$$

where we have used the vector notation $X' = \frac{dX}{dT}$, $P' = \frac{dP}{dT}$. By the product rule and the fact that equations (3.2) are valid for non-vanishing f_i and g_i, this expression reduces to

$$\left[\tilde{U}^{(1)}\left(\frac{dX_i}{dT} - \{X_i, K\}\right)\right]\Bigg|_{(3.2)} = 0$$

$$\left[\tilde{U}^{(1)}\left(\frac{dP_i}{dT} - \{P_i, K\}\right)\right]\Bigg|_{(3.2)} = 0 \;. \qquad (3.5)$$

These equations are precisely the Lie determining equations for the transformed equations of motion everywhere the f_i and g_i are non-zero. Notice also that by virtue of the method of construction of $\tilde{U}^{(1)}$, the commutation relations of a set of Lie generators of the form of $U^{(1)}$ are preserved under this change of variables. Thus it is seen that one may derive a different realization of a given symmetry algebra of a Hamiltonian system if one has a system covariance transformation of that Hamiltonian system (for a definition of a system covariance transformation, see Appendix I).

CHAPTER IV

REVIEW OF KNOWN RESULTS FOR THE FREE PARTICLE

For the free particle problem with N degrees of freedom and mass m, the Hamiltonian is $H(x,p,t) = p_i p_i/2m$ and the equations of motion are

$$\dot{x}_i = p_i/m$$
$$\dot{p}_i = 0 . \qquad (4.1)$$

The Lie determining equations for these equations of motion are

$$m\,\dot{\eta}^{x_i} - p_i\,\dot{\xi} = \eta^{p_i}$$
$$\dot{\eta}^{p_i} = 0 . \qquad (4.2)$$

We notice immediately that

$$\eta^{x_i} = m^{-1} a_i t + b_i$$
$$\eta^{p_i} = a_i \qquad (4.3)$$
$$\xi = c ,$$

where a_i, b_i, and c are constants of the motion (or arbitrary differentiable functions of them), is a solution of these determining equations.

We reproduce here some previously known but unpublished facts about the maximum space-time transformation group of the free particle problem with N degrees of freedom[11]. In reference [11] it was shown that the solution of equations (4.2) when one restricts ξ and η^{x_i} to be functions of x and t only implies the following set of $(N+2)^2 - 1$ free particle symmetry generators:

$$\begin{aligned}
U_{00} &= t\,\partial_t - p_i\,\partial_{p_i} \\
U_{i0} &= x_i\,\partial_t - m^{-1} p_i p_j\,\partial_{p_j} \\
U_{0i} &= t\,\partial_{x_i} + m\,\partial_{p_i} \\
U_{ij} &= x_i\,\partial_{x_j} + p_i\,\partial_{p_j} \\
U_{N+1,0} &= \partial_t \\
U_{0,N+1} &= -t^2\partial_t - t x_i\,\partial_{x_i} + (-m x_i + p_i t)\,\partial_{p_i} \\
U_{N+1,i} &= \partial_{x_i} \\
U_{i,N+1} &= -t x_i\,\partial_t - x_i x_j\,\partial_{x_j} - p_i(x_j - m^{-1} p_j t)\,\partial_{p_j}\,.
\end{aligned} \qquad (4.4)$$

It was noted in reference [11] that these generators have the same commutation relations as the Lie algebra sl(N+2,R) when one introduces the additional operator

$$U_{N+1,N+1} = -\sum_{\alpha=0}^{N+1} U_{\alpha\alpha}\,.$$

Thus it was found that

$$[U_{\alpha\beta}, U_{\gamma\rho}] = \delta_{\beta\gamma} U_{\alpha\rho} - \delta_{\alpha\rho} U_{\gamma\beta}, \quad (4.5)$$

where α, β, γ, $\rho = 0, 1, \ldots, N, N+1$ and $\delta_{\alpha\beta}$ is the Kronecker delta.

An important observation made in reference [11] is that the transformation group SL(N+2,R) (as realized by these generators and their finite transformations) is a transitive symmetry transformation group acting on the solution space of the free particle problem with N degrees of freedom; i.e., any free particle solution can be obtained from any other by an application of an element of the transformation group.

To see this, one only needs to examine the effect of the finite transformations corresponding to the generators $U_{N+1,0}$, $U_{N+1,i}$, and U_{0i} on a free particle solution; notice that these are the generators implied by the solution (4.3) of the determining equations. For completeness, all of the finite transformations with their corresponding generators are listed here (a corresponds to a group parameter):

$$U_{00}: \quad t' = at, \quad x_i' = x_i, \quad p_i' = p_i/a \qquad\qquad U_{i0}: \quad t' = t + a x_i, \quad x_i' = x_i, \quad p_j' = p_j[1 + a m^{-1} p_i]^{-1} \quad (4.6)$$

$$U_{0i}: \quad t' = t, \quad x_j' = x_j + at\delta_{ij}, \quad p_j' = p_j + am\,\delta_{ij}$$

$$U_{ij}: \quad t' = t, \quad x_k' = x_k + a x_i \delta_{jk}, \quad p_k' = p_k + a p_i \delta_{jk}$$

$$(4.6)$$

$$U_{N+1,0}: \quad t' = t + a, \quad x_i' = x_i, \quad p_i' = p_i$$

$$U_{N+1,i}: \quad t' = t, \quad x_j' = x_j + a\delta_{ij}, \quad p_i' = p_i$$

$$U_{0,N+1}: \quad t' = t[1+at]^{-1}, \quad x_i' = x_i[1+at]^{-1}, \quad p_i' = p_i - a(mx_i - p_i t)$$

$$U_{i,N+1}: \quad t' = t[1+ax_i]^{-1}, \quad x_j' = x_j[1+ax_i]^{-1}, \quad p_j' = \frac{p_j + a(x_i p_j - x_j p_i)}{1 + a(x_i - m^{-1} p_i t)}$$

CHAPTER V

POISSON OPERATORS AND CONSTANTS OF THE MOTION

This chapter establishes an algorithmic connection between some of the constants of the motion of a given Hamiltonian system and some of the Lie symmetry generators of that system. When this connection exists, it provides an alternative to Noether's theorem[2] for finding a conserved quantity of a physical system when one knows a symmetry of the mathematical formulation.

We first show that if we have a constant of the motion A(x,p,t) of a Hamiltonian system described by the Hamiltonian function H(x,p,t) then the Poisson operator $\{\cdot,A\}$ is a Lie generator of the system. The proof is straightforward. By definition, we have

$$\dot{A}(x,p,t) = \{A,H\} + \partial_t A \equiv 0 \qquad (5.1)$$

and if we make the identification

$$\eta^{x_i} = A_{p_i} \quad , \quad \eta^{p_i} = -A_{x_i}$$

then the left hand side of the determining equation (1.6a) becomes

$$\{A_{p_i}, H\} + A_{tp_i} - H_{p_i x_j} A_{p_j} - H_{p_i p_j} A_{x_j}$$

$$= \{A_{p_i}, H\} + \{A, H_{p_i}\} + A_{tp_i}$$

$$= \partial_{p_i} \{A, H\} + A_{tp_i} .$$

From equation (5.1) we know that $\{A,H\} \equiv -\partial_t A$. Hence this expression reduces to zero. The proof for the other determining equation is entirely analogous; thus the statement is proved.

We note immediately that the constant of the motion A(x,p,t) is an invariant function under the transformation group element generated by $\{\cdot,A\}$; this is because $\{\cdot,A\}A \equiv 0$, which is a necessary and sufficient condition for the invariance of A(x,p,t).

If we have a Lie symmetry generator of the form

$$L = \eta^{x_i}(x,p,t)\,\partial_{x_i} + \eta^{p_i}(x,p,t)\,\partial_{p_i} \tag{5.2}$$

and if the N^2 integrability conditions

$$\eta^{x_i}_{x_j} = -\eta^{p_i}_{p_j} \tag{5.3}$$

are satisfied, then there is a constant of the motion A(x,p,t) associated with L. We find A(x,p,t) by integrating the 2N equations

$$A_{x_i} = -\eta^{p_i}$$

$$A_{p_i} = \eta^{x_i} . \qquad (5.4)$$

To prove this, we note that if the integrability conditions (5.3) hold, then the function A(x,p,t) exists. We can write the identity $-A_{p_i t} + A_{tp_i} = 0$ as $\partial_{p_i}\{A,H\} + A_{tp_i} = 0$ only if we have $\{A,H\} \equiv -\partial_t A$; i.e., the function A must satisfy equation (5.1). Thus in order to go backwards through the proof of the previous page (which we must be able to do since $\{\cdot,A\}$ was assumed to be a Lie symmetry generator), the function A must be a constant of the motion. Equations (5.4) determine A only up to an arbitrary additive function of t; equation (5.1), which we now know must hold, then shows that this additive function of t must be a constant.

An example is provided by the free particle Lie symmetry generator $U_{ij} = x_i\partial_{x_j} + p_i\partial_{p_j}$. This generator does not satisfy the integrability conditions (5.3). However, if we take the linear combination $U_{ij} - U_{ji}$ and rewrite it as

$$U_{ij} - U_{ji} = (x_i\delta_{jk} - x_j\delta_{ik})\partial_{x_k} + (p_i\delta_{jk} - p_j\delta_{ik})\partial_{p_k} ,$$

we find by the above technique the corresponding free particle constant of the motion $x_ip_j - x_jp_i$ to within an arbitrary additive constant.

These results also apply to the free particle constants of the motion $x_i - tp_i/m$ and p_i, which correspond to the Poisson bracket operators U_{0i} and $U_{N+1,i}$, respectively, of the set (4.4) which we already know to be free particle Lie symmetry generators. Similar considerations apply to the harmonic oscillator constants of the motion D_i and E_i and the harmonic oscillator symmetry generators $V_{N+1,i}$ and V_{0i}, respectively, presented in chapter VI.

Because of the results presented in chapters II and III and references [4] and [11], we note that the above free particle and harmonic oscillator Lie symmetry generators, which are Poisson operators of constants of the motion, integrate to group elements of an SL(N+2,R) transformation group for each problem. For the connection with general dynamical systems, see reference [4].

It may be possible to simplify a given Lie symmetry generator of the Poisson operator form. The Poisson operator of the function $F(A_1,\ldots,A_m)$ of the constants of the motion A_μ, $\mu = 1,\ldots,m$ can be written

$$\{\cdot, F\} = \frac{\partial F}{\partial A_\mu}\frac{\partial A_\mu}{\partial p_i}\partial_{x_i} - \frac{\partial F}{\partial A_\mu}\frac{\partial A_\mu}{\partial x_i}\partial_{p_i} = \frac{\partial F}{\partial A_\mu}\{\cdot, A_\mu\} .$$

Thus $\{\cdot,F\}$ is an unspecified (if the trajectory is not given) linear combination of the Poisson operators $\{\cdot,A_\mu\}$, each of which are, in general, simpler in form than $\{\cdot,F\}$.

We note from the commutation relations (4.5) that U_{ij} acts as a raising and lowering operator for U_{0i} and $U_{N+1,i}$:

$$[U_{0i}, U_{ij}] = U_{0j}$$

$$[U_{N+1,i}, U_{ij}] = U_{N+1,j} .$$

This property can be useful calculationally.

The connection of these results with the work of Lie on invariant functions and path curves[13] is presently under study.

CHAPTER VI

EXPLICIT EXAMPLES

1. Free Particle to Harmonic Oscillator

In this chapter we use the solution transformation (2.5) and the Bluman transformation (2.6) to produce different realizations of particular transformation groups in terms of free particle and isotropic harmonic oscillator variables. For convenience, we reproduce these transformations here. Free particle variables are denoted by x_i and p_i; harmonic oscillator variables are are denoted by X_i and P_i. When the parameter t is transformed, its transformed value is denoted by T. We will, however, write all generators and transformations in terms of the variables x_i, p_i, and t. Free particle generators will be denoted generically by U and harmonic oscillator generators will be denoted generically by V.

The solution transformation is

$$\begin{aligned} x_i &= \alpha X_i - \omega^{-1}\beta P_i \\ p_i &= s X_i + \omega^{-1} c P_i \end{aligned} \qquad (6.1)$$

where $s = \sin\omega t$, $c = \cos\omega t$, $\alpha = c + ts$, and $\beta = s - tc$ (this somewhat unconventional notation is very convenient for calculational purposes). The inverse of this transformation is

$$\begin{aligned} X_i &= c x_i + \beta \rho_i \\ P_i &= -\omega s x_i + \omega \alpha \rho_i . \end{aligned} \tag{6.2}$$

The Bluman transformation of chapter II is

$$\begin{aligned} t &= \tan \omega T \\ x_i &= X_i \sec \omega T \\ \rho_i &= X_i \sin \omega T + \omega^{-1} P_i \cos \omega T . \end{aligned} \tag{6.3}$$

The inverse of this transformation is

$$\begin{aligned} T &= \omega^{-1} \tan^{-1} t \\ X_i &= x_i A \\ P_i &= -\omega t x_i A + \omega A^{-1} \rho_i \end{aligned} \tag{6.4}$$

where $A = (1 + t^2)^{1/2}$.

The free particle Hamiltonian equations of motion are

$$\begin{aligned} \dot{x}_i &= \rho_i / m \\ \dot{\rho}_i &= 0 \end{aligned} \tag{6.5}$$

and the determining equations corresponding to these are

$$\begin{aligned} m \dot{\eta}^{x_i} - \rho_i \dot{\xi} &= \eta^{\rho_i} \\ \dot{\eta}^{\rho_i} &= 0 . \end{aligned} \tag{6.6}$$

The harmonic oscillator equations of motion are

$$\begin{aligned} \dot{x}_i &= \rho_i \\ \dot{\rho}_i &= -\omega^2 x_i \end{aligned} \tag{6.7}$$

and the determining equations corresponding to these equations of motion are

$$\begin{aligned} \dot{\eta}^{x_i} - \rho_i \dot{\xi} &= \eta^{\rho_i} \\ \dot{\eta}^{\rho_i} + \omega^2 x_i \dot{\xi} &= -\omega^2 \eta^{x_i} . \end{aligned} \tag{6.8}$$

If one applies the solution transformation (6.1) to the set of free particle generators (4.4), one obtains the following realization of sl(N+2,R) in terms of harmonic oscillator variables (we choose m = 1 in equations (4.4)):

$$\begin{aligned}
V_{00} &= t\partial_t + [-s^2 x_i + (t-\omega^{-1}sc)\rho_i]\partial_{x_i} - [\omega(\omega t + sc)x_i + c^2\rho_i]\partial_{\rho_i} \\
V_{i0} &= [\alpha x_i - \omega^{-1}\beta\rho_i]\partial_t - [s x_i x_j + (\omega^{-1}c - \alpha)x_i\rho_j + \omega^{-1}\beta\rho_i\rho_j]\partial_{x_j} \\
&\quad - [\omega^2\alpha x_i x_j + (s-\omega\beta)x_j\rho_i + \omega^{-1}c\rho_i\rho_j]\partial_{\rho_j} \\
V_{0i} &= s\partial_{x_i} + \omega c\partial_{\rho_i} \\
V_{N+1,0} &= \partial_t + [-sc x_i + \omega^{-1}(\omega - c^2)\rho_i]\partial_{x_i} + [\omega(s^2-\omega)x_i + sc\rho_i]\partial_{\rho_i} \\
V_{0,N+1} &= -t^2\partial_t + [-sc x_i + (\omega^{-1}s^2 - t^2)\rho_i]\partial_{x_i} \\
&\quad + [\omega(\omega t^2 - c^2)x_i + sc\rho_i]\partial_{\rho_i} \\
V_{N+1,i} &= c\partial_{x_i} - \omega s\partial_{\rho_i} \quad , \quad V_{ij} = x_i\partial_{x_j} + \rho_i\partial_{\rho_j}
\end{aligned} \tag{6.9}$$

$$V_{i,N+1} = [-\alpha t x_i + \omega^{-1}\beta t \rho_i]\partial_t + [-c x_i x_j + (\omega^{-1}s - \alpha t)x_i\rho_j + \omega^{-1}\beta t \rho_i\rho_j]\partial_{x_j} + [\omega^2 \alpha t x_i x_j - (c + \omega\beta t)x_j\rho_i + \omega^{-1}s\rho_i\rho_j]\partial_{\rho_j}$$

where, as before, $s = \sin\omega t$, $c = \cos\omega t$, $\alpha = c + ts$, and $\beta = s - tc$. These generators satisfy the harmonic oscillator determining equations (6.8).

Similarly, one may transform the set of free particle generators (4.4) by the Bluman transformation (6.3) to find the following realization of sl(N+2,R) in terms of harmonic oscillator variables:

$$V'_{00} = \omega^{-1}sc\,\partial_t - s^2 x_i\,\partial_{x_i} - [2\omega s c x_i + c^2\rho_i]\partial_{\rho_i}$$

$$V'_{i0} = \omega^{-1}c x_i\,\partial_t - s x_i x_j\,\partial_{x_j} - [\omega c x_i x_j + s x_j\rho_i + \omega^{-1}c\rho_i\rho_j]\partial_{\rho_j}$$

$$V'_{0i} = s\partial_{x_i} + \omega c\,\partial_{\rho_i} \tag{6.10}$$

$$V'_{ij} = x_i\partial_{x_j} + \rho_i\partial_{\rho_j}$$

$$V'_{N+1,0} = \omega^{-1}c^2\partial_t - sc x_i\,\partial_{x_i} + [\omega(s^2 - c^2)x_i + sc\rho_i]\partial_{\rho_i}$$

$$V'_{0,N+1} = -\omega^{-1}s^2\partial_t - sc x_i\,\partial_{x_i} + [\omega(s^2 - c^2)x_i + sc\rho_i]\partial_{\rho_i}$$

$$V'_{N+1,i} = c\,\partial_{x_i} - \omega s\,\partial_{\rho_i}$$

$$V_{i,N+1} = -\omega^{-1} s x_i \partial_t - c x_i x_j \partial_{x_j} + [\omega s x_i x_j - c x_j \rho_i + \omega^{-1} s \rho_i \rho_j] \partial_{\rho_j}$$

where, as before, $s = \sin\omega t$ and $c = \cos\omega t$. These generators satisfy the harmonic oscillator determining equations (6.8). We note here that these generators are the maximal space-time set of Lie symmetry generators for the isotropic harmonic oscillator problem with N degrees of freedom (ξ and the η^{x_i} depend only on x and t). The generators for the one degree of freedom case were derived by R.L. Anderson and discussed by Wulfman and Wybourne[12].

2. Harmonic Oscillator to Free Particle

One may take the above two sets of generators and transform them back to generators in terms of free particle variables. The set (6.9), which we denote by {V}, is transformed by the inverse of the Bluman transformation into the set {U'} and the set (6.10), which we denote by {V'}, is transformed by the inverse of the solution transformation into the set {U''}. If we denote the solution transformation by S and the Bluman transformation by B, we have

$$\begin{array}{lll} & \xrightarrow{S} \{V\} \xrightarrow{B^{-1}} \{U'\} & , \quad \{U\} \neq \{U'\} \neq \{U''\} \\ \{U\} & & \\ & \xrightarrow{B} \{V'\} \xrightarrow{S^{-1}} \{U''\} & , \quad \{V\} \neq \{V'\} \end{array}$$

Transforming the set of generators {V} by the inverse of the Bluman transformation (B^{-1}), we obtain the following set {U'} of free particle generators (here we have chosen $m = 1$ and $\omega = 1$):

$$
\begin{aligned}
U'_{00} &= \lambda\,\partial_t + \rho_i(\lambda - t)\,\partial_{x_i} - \rho_i\,\partial_{\rho_i} \\
U'_{i0} &= \left[(1+t^2)\left(x_i - \rho_i(t - \tan^{-1}t)\right)\right]\partial_t \\
&\quad + \left[t^2 x_i \rho_j - (1+t^2)(t - \tan^{-1}t)\rho_i\rho_j\right]\partial_{x_j} - \rho_i\rho_j\,\partial_{\rho_j} \\
U'_{0i} &= t\,\partial_{x_i} + \partial_{\rho_i} \\
U'_{ij} &= x_i\,\partial_{x_j} + \rho_i\,\partial_{\rho_j} \qquad\qquad (6.11) \\
U'_{N+1,0} &= (1+t^2)\,\partial_t + t^2\rho_i\,\partial_{x_i} \\
U'_{0,N+1} &= -\lambda\tan^{-1}t\,\partial_t - \left[t(x_i - \rho_i t) + \lambda\rho_i\tan^{-1}t\right]\partial_{x_i} \\
&\quad - (x_i - \rho_i t)\,\partial_{\rho_i} \\
U_{N+1,i} &= \partial_{x_i} \\
U_{i,N+1} &= \left[-\lambda x_i + \lambda(t - \tan^{-1}t)\rho_i\right]\partial_t \\
&\quad + \left[-x_i x_j + (t-\lambda)x_i\rho_j + \lambda(t - \tan^{-1}t)\rho_i\rho_j\right]\partial_{x_j} \\
&\quad - \rho_i(x_j - \rho_j t)\,\partial_{\rho_j}
\end{aligned}
$$

where $\lambda = (1 + t^2)\tan^{-1}t$.

These generators solve the free particle determining equations (6.6); they are distinct and linearly independent of the set {U} (equations (4.4)).

Transforming the set of generators {V'} by the inverse of the solution transformation (S^{-1}), we obtain the following set {U''} of free particle generators (again, we have chosen $m = 1$ and $\omega = 1$):

$$
\begin{aligned}
U''_{00} &= sc\,\partial_t + (sc-t)\rho_i\,\partial_{x_i} - \rho_i\,\partial_{\rho_i} \\
U''_{i0} &= [c^2 x_i + \beta c \rho_i]\,\partial_t + [-s^2 x_i\rho_j + \beta c\,\rho_i\rho_j]\,\partial_{x_j} - \rho_i\rho_j\,\partial_{\rho_j} \\
U''_{0i} &= t\,\partial_{x_i} + \partial_{\rho_i} \\
U''_{ij} &= x_i\,\partial_{x_j} + \rho_i\,\partial_{\rho_j} \qquad\qquad (6.12) \\
U''_{N+1,0} &= c^2\partial_t - s^2\rho_i\,\partial_{x_i} \\
U''_{0,N+1} &= -s^2\partial_t + c^2\rho_i\,\partial_{x_i} \\
U''_{N+1,i} &= \partial_{x_i} \\
U''_{i,N+1} &= [-sc\,x_i - \beta s\,\rho_i]\partial_t + [-x_i x_j + (t-sc)\,x_i\rho_j - \beta s\,\rho_i\rho_j]\,\partial_{x_j} \\
&\quad - \rho_i\,(x_j - \rho_j t)\,\partial_{\rho_j}
\end{aligned}
$$

where, as before, $s = \sin\omega t$, $c = \cos\omega t$, and $\beta = s - tc$ (here

$\omega = 1$). These generators solve the free particle determining equations (6.6); they are distinct and linearly independent of the sets {U} and {U'} (equations (4.4) and (6.11), respectively).

Thus we see that, including the fundamental equations (4.4), we have five distinct realizations of the Lie algebra sl(N+2,R). One could continue the alternation of the solution transformation S and the Bluman transformation B to obtain other realizations.

3. Realizations of Mukunda's Generators for o(N+2,R)

In this section, we obtain realizations of an o(N+2) Lie algebra of Mukunda[13] in terms of free particle and harmonic oscillator variables. Mukunda showed that if $\dot{X}_i = 0$ and $\dot{P}_i = 0$ for a given Hamiltonian system (i.e., X_i and P_i are constants of the motion), then the following functions close under the Poisson bracket to form a realization of the Lie algebra o(N+2):

$$\tilde{J}_{RS} = X_R P_S - X_S P_R \quad , \quad \tilde{S} = m^{-1} F'' G''$$

$$\tilde{A}_R = G'' X_R \quad \text{where} \quad G'' = (m^2 - P_i P_i)^{1/2}$$

$$\tilde{B}_R = m^{-1} F'' P_R \quad \text{where} \quad F'' = (\alpha^2 - m^2 X_i X_i + (X_i P_i)^2)^{1/2} ,$$

where m and α are constants.

It is not difficult to show that if a set of functions $A_i(x,p,t)$ form a realization of a Lie algebra with the Lie bracket being the Poisson bracket, then the operators $\{\cdot, A_i\}$ form a realization of the same Lie algebra where the Lie bracket is now the commutator[7]. If we choose the free particle as an example and let $X_i = x_i - p_i t$ and $P_i = p_i$, then by this method of construction we derive the following operators:

$$J_{RS} = x_s \partial_{x_R} - x_R \partial_{x_s} + p_s \partial_{p_R} - p_R \partial_{p_s} \tag{6.13}$$

$$A_R = [G'^{-1}{}_{p_i}(x_R - p_R t) + \delta_{iR} G' t]\partial_{x_i} + \delta_{iR} G' \partial_{p_i}$$

$$B_R = m^{-1}[-m^2 F'^{-1}(x_i - p_i t)p_R t - F'^{-1}(x_i - 2p_i t)(x_j - p_j t)p_j p_R + - \delta_{iR} F']\partial_{x_i} + m^{-1} F'^{-1}{}_{p_R}[-m^2(x_i - p_i t) + (x_j - p_j t)p_j p_i]\partial_{p_i}$$

$$S = -m^{-1}[G' F'^{-1} m^2 (x_i - p_i t)t + G' F'^{-1}(x_i - 2p_i t)(x_j - p_j t)_{p_j} - F' G'^{-1}{}_{p_i}]\partial_{x_i} + m^{-1} G' F'^{-1}[-m^2(x_i - p_i t) + (x_j - p_j t)p_j p_i]\partial_{p_i}$$

where

$$G' = G''\Big|_{\substack{X_i = x_i - \rho_i t \\ P_i = \rho_i}} \quad , \quad F' = F''\Big|_{\substack{X_i = x_i - \rho_i t \\ P_i = \rho_i}} \quad .$$

These generators solve the free particle determining equations (6.6); hence they are free particle symmetry generators. This could have been seen directly from the considerations given in chapter V.

We now write the set of generators (6.13) in terms of harmonic oscillator variables. Applying the solution transformation (6.1) to the set of generators (6.13), we derive the following set of generators:

$$J'_{RS} = x_S \partial_{x_R} - x_R \partial_{x_S} + \rho_S \partial_{\rho_R} - \rho_R \partial_{\rho_S} \qquad (6.14)$$

$$A'_R = \left[G^{-1}E_R D_i c + \delta_{iR} G s\right]\partial_{x_i} + \left[-G^{-1}E_R D_i s + \delta_{iR} D c\right]\partial_{\rho_i}$$

$$B'_R = m^{-1}\left[F^{-1}E_j D_j D_R(-E_i c + D_i s) - m^2 F^{-1} E_i D_R s - \delta_{iR} F c\right]\partial_{x_i}$$
$$+ m^{-1}\left[F^{-1}E_j D_j D_R(E_i s + D_i c) - m^2 F^{-1} E_i D_R c + \delta_{iR} F s\right]\partial_{\rho_i}$$

$$S' = \left[G F^{-1} m^{-1}\left(-E_i E_j D_j c - m^2 E_i s + E_j D_j D_i s\right) + m^{-1} F G^{-1} D_i c\right]\partial_{x_i}$$
$$+ \left[G F^{-1} m^{-1}\left(E_i E_j D_j s - m^2 E_i c + E_j D_j D_i c\right) - m^{-1} F G^{-1} D_i s\right]\partial_{\rho_i}$$

where $s = \sin\omega t$, $c = \cos\omega t$, (here, $\omega = 1$), $E_i = x_i c - p_i s$,

$D_i = x_i s + p_i c$, $G = (m^2 - D_i D_i)^{1/2}$, and

$$F = (\alpha^2 - E_i E_i + (E_i D_i)^2)^{1/2}.$$

The use of the somewhat superfluous notation is to emphasize that F, G, D_i, and E_i are harmonic oscillator constants of the motion. These generators solve the harmonic oscillator determining equations (6.8) and thus they are harmonic oscillator symmetry generators which close under commutation to yield a realization of the Lie algebra o(N+2).

4. Finite transformations

In this section we give the finite transformations corresponding to the sets of infinitesimal generators {V} and {V'} (equations (6.9) and (6.10), respectively). In general, it is extremely difficult to obtain these finite transformations from the infinitesimal generators, either by exponentiation (when it has meaning) or by solving the corresponding Lie differential equations. Based on our possession of the free particle finite transformations (4.6), we use an alternate method for finding these finite transformations (see the acknowledgement to A. O. Barut in reference [4]).

Suppose we have a one parameter group of transformations for a Hamiltonian system Σ_1 (a corresponds to a group parameter):

$$
\begin{aligned}
x_i' &= A^i(x,p,t\,;a) \\
p_i' &= B^i(x,p,t\,;a) \qquad\qquad (6.15) \\
t' &= C(x,p,t\,;a)
\end{aligned}
$$

and suppose we also have a transformation connecting system Σ_1 with another Hamiltonian system Σ_2 with the same number of degrees of freedom as system Σ_1:

$$
\begin{aligned}
T &= \Theta(x,p,t) & & & t &= \theta(X,P,T) \\
X_i &= \Phi^i(x,p,t) & \text{OR} & & x_i &= \phi^i(X,P,T) \qquad (6.16) \\
P_i &= \Psi^i(x,p,t) & & & p_i &= \psi^i(X,P,T)\,.
\end{aligned}
$$

To obtain the finite transformations for the system Σ_2, we note that we must certainly also have

$$
\begin{aligned}
T' &= \Theta(x',p',t') \\
X_i' &= \Phi^i(x',p',t') \\
P_i' &= \Psi^i(x',p',t')\,.
\end{aligned}
$$

Then using equations (6.15) and (6.16), we can write this in terms of the variables X_i, P_i, and T:

$$T' = \Theta\big(A(\phi,\psi,\theta;a),\, B(\phi,\psi,\theta;a),\, C(\phi,\psi,\theta;a)\big)$$

$$X_i' = \Phi^i\big(A(\phi,\psi,\theta;a),\, B(\phi,\psi,\theta;a),\, C(\phi,\psi,\theta;a)\big) \qquad (6.17)$$

$$P_i' = \Psi^i\big(A(\phi,\psi,\theta;a),\, B(\phi,\psi,\theta;a),\, C(\phi,\psi,\theta;a)\big)$$

where we have used vector notation. These are the one parameter group transformations of system Σ_2 corresponding to the one parameter group transformations of system Σ_1.

By using this technique, one obtains the following set of finite transformations corresponding to the set {V} (equations (6.9)) of harmonic oscillator generators (a corresponds to a group parameter):

V_{00}:

$$t' = at \qquad (6.18a)$$

$$x_i' = [c\hat{c} + a^{-1}s\hat{s}]x_i + \omega^{-1}[-s\hat{c} + a^{-1}c\hat{s}]p_i$$

$$p_i' = \omega[-\hat{s}c + a^{-1}s\hat{c}]x_i + [s\hat{s} + a^{-1}c\hat{c}]p_i$$

where $s = \sin\omega t$, $c = \cos\omega t$, $\hat{s} = \sin a\omega t$, and $\hat{c} = \cos a\omega t$. Note that this reduces to the identity transformation for $a = 1$.

V_{i0}:

$$t' = t + a(\alpha x_i - \omega^{-1}\beta p_i) \qquad (6.18b)$$

$$x_j' = (\alpha x_j - \omega^{-1}\beta\rho_j)\cos\omega\left(t + a(\alpha x_i - \omega^{-1}\beta\rho_i)\right) +$$

$$+ \frac{s x_j + \omega^{-1} c\rho_j}{1 + a(s x_i + \omega^{-1} c\rho_i)}\Big[\sin\omega\left(t + a(\alpha x_i - \omega^{-1}\beta\rho_i)\right) +$$

$$- \left(t + a(\alpha x_i - \omega^{-1}\beta\rho_i)\right)\cos\omega\left(t + a(\alpha x_i - \omega^{-1}\beta\rho_i)\right)\Big]$$

$$(6.18b)$$

$$\rho_j' = -\omega(\alpha x_j - \omega^{-1}\beta\rho_j)\sin\omega\left(t + a(\alpha x_i - \omega^{-1}\beta\rho_i)\right) +$$

$$+ \frac{\omega s x_j + c\rho_j}{1 + a(s x_i + \omega^{-1} c\rho_i)}\Big[\cos\omega\left(t + a(\alpha x_i - \omega^{-1}\beta\rho_i)\right) +$$

$$+ \left(t + a(\alpha x_i - \omega^{-1}\beta\rho_i)\right)\sin\omega\left(t + a(\alpha x_i - \omega^{-1}\beta\rho_i)\right)\Big].$$

Again, $s = \sin\omega t$, $c = \cos\omega t$, $\alpha = c + ts$, and $\beta = s - tc$. This reduces to the identity transformation when $a = 0$.

V_{0i}:

$$t' = t$$
$$x_j' = x_j + a\delta_{ij}\sin\omega t \qquad (6.18c)$$
$$\rho_j' = \rho_j + a\delta_{ij}\,\omega\cos\omega t$$

V_{ij}:

$$t' = t$$
$$x_k' = x_k + a x_i \delta_{jk} \qquad (6.18d)$$
$$\rho_k' = \rho_k + a\rho_i\delta_{jk}$$

$V_{N+1,0}$:

$$t' = t + a$$

$$x_i' = [\alpha\hat{c} + \hat{\beta}s]x_i + \omega^{-1}[\hat{\beta}c - \beta\hat{c}]\rho_i$$

$$\rho_i' = \omega[s\hat{\alpha} - \hat{s}\alpha]x_i + [\beta\hat{s} + \hat{\alpha}c]\rho_i \tag{6.18e}$$

where $\hat{s} = \sin\omega\tau$, $s = \sin\omega t$, $\hat{c} = \cos\omega\tau$, $c = \cos\omega t$, $\hat{\alpha} = \hat{c} + \tau\hat{s}$, $\hat{\beta} = \hat{s} - \tau\hat{c}$, $\alpha = c + ts$, $\beta = s - tc$, and $\tau = t + a$. Note that this gives the identity transformation for $a = 0$.

$V_{0,N+1}$:

$$t' = t[1 + at]^{-1}$$

$$x_i' = [c\hat{c} + \hat{s}(s - ac)]x_i + \omega^{-1}[-sc + \hat{s}(c + as)]\rho_i$$

$$\rho_i' = \omega[s\hat{c} - c(\hat{s} + a\hat{c})]x_i + [s\hat{s} + \hat{c}(c + as)]\rho_i \tag{6.18f}$$

where $\hat{c} = \cos[\omega t(1+at)^{-1}]$, $\hat{s} = \sin[\omega t(1+at)^{-1}]$, $s = \sin\omega t$, and $c = \cos\omega t$. This reduces to the identity transformation when $a = 0$.

$V_{N+1,i}$:

$$t' = t$$

$$x_j' = x_j + a\delta_{ij}\cos\omega t \tag{6.18g}$$

$$\rho_j' = \rho_j - a\omega\delta_{ij}\sin\omega t$$

$$V_{i,N+1}: \quad t' = At$$

$$x_j' = \hat{c}\,A\,(\alpha x_j - \omega^{-1}\beta p_j) + $$

$$+ (\hat{s} - At\hat{c})\left[\frac{s x_j + \omega^{-1} c p_j + a\omega^{-1}(x_i p_j - x_j p_i)}{1 + a(c x_i - \omega^{-1} s p_i)}\right]$$

$$p_j' = -\omega\hat{s}\,A\,(\alpha x_j - \omega^{-1}\beta p_j) + \tag{6.18h}$$

$$+ \omega(\hat{c} + At\hat{s})\left[\frac{s x_j + \omega^{-1} c p_j + a\omega^{-1}(x_i p_j - x_j p_i)}{1 + a(c x_i - \omega^{-1} s p_i)}\right]$$

where $A = [1 + a(\alpha x_i - \omega^{-1}\beta p_i)]^{-1}$, $\hat{s} = \sin A\omega t$, $\hat{c} = \cos A\omega t$, $s = \sin\omega t$, $c = \cos\omega t$, $\alpha = c + ts$, and $\beta = s - tc$. Note that this reduces to the identity transformation for $a = 0$.

We also write down the finite transformations corresponding to the set $\{V'\}$ (equations (6.10)) of harmonic oscillator generators (a corresponds to a group parameter):

$$V_{00}': \quad t' = \omega^{-1}\tan^{-1}(a\tan\omega t)$$

$$x_i' = \frac{x_i \sec\omega t}{\sqrt{1 + a^2\tan^2\omega t}} \tag{6.19a}$$

$$p_i' = \omega x_i\left[a^{-1}\sqrt{1 + a^2\tan^2\omega t}\,\sin\omega t - \frac{a\tan\omega t\sec\omega t}{\sqrt{1 + a^2\tan^2\omega t}}\right]$$

$$+ \left[a^{-1}\sqrt{1 + a^2\tan^2\omega t}\,\cos\omega t\right] p_i \, .$$

Note that this produces the identity transformation for $|a| = 1$ and we must have $|a| > 0$.

$$V'_{io}: \quad t' = \omega^{-1}\tan^{-1}\left[\tan\omega t + a x_i \sec\omega t\right]$$

$$x'_j = \frac{x_j \sec\omega t}{\sqrt{1+\left[\tan\omega t + a x_i \sec\omega t\right]^2}} \qquad (6.19b)$$

$$\rho'_j = \frac{-\omega\left[\tan\omega t + a x_i \sec\omega t\right] x_j \sec\omega t}{\sqrt{1+\left[\tan\omega t + a x_i \sec\omega t\right]^2}} +$$

$$+ \omega\sqrt{1+\left[\tan\omega t + a x_i \sec\omega t\right]^2}\left[\frac{x_j \sin\omega t + \omega^{-1}\rho_j \cos\omega t}{1 + a\left[x_i \sin\omega t + \omega^{-1}\rho_i \cos\omega t\right]}\right].$$

We see this yields the identity transformation for a = 0.

$$V'_{oi}: \quad t' = t$$

$$x'_j = x_j + a\,\delta_{ij} \sin\omega t \qquad (6.19c)$$

$$\rho'_j = \rho_j + a\omega\,\delta_{ij} \cos\omega t$$

$$V'_{ij}: \quad t' = t$$

$$x'_k = x_k + a x_i \delta_{jk} \qquad (6.19d)$$

$$\rho'_k = \rho_k + a \rho_i \delta_{jk}$$

$$V'_{N+1,0}: \quad t' = \omega^{-1}\tan^{-1}(a + \tan\omega t)$$

$$x_i' = \frac{x_i \sec\omega t}{\sqrt{1+(a+\tan\omega t)^2}} \qquad (6.19e)$$

$$p_i' = x_i \left[x \sin\omega t \sqrt{1+(a+\tan\omega t)^2} - \frac{\omega(a+\tan\omega t)\sec\omega t}{\sqrt{1+(a+\tan\omega t)^2}} \right]$$

$$+ p_i \cos\omega t \sqrt{1+(a+\tan\omega t)^2} \,,$$

We again see that we get the identity transformation for a = 0.

$$V'_{0,N+1}: \quad t' = \omega^{-1}\tan^{-1}\left(\frac{\tan\omega t}{1+a\tan\omega t}\right)$$

$$x_i' = \frac{x_i \sec\omega t}{\sqrt{(1+a\tan\omega t)^2 + \tan^2\omega t}} \qquad (6.19f)$$

$$p_i' = \frac{-\omega x_i \tan\omega t \sec\omega t}{\sqrt{(1+a\tan\omega t)^2+\tan^2\omega t}} +$$

$$+ \omega \sqrt{1+\left(\frac{\tan\omega t}{1+a\tan\omega t}\right)^2} \left[x_i \sin\omega t + \omega^{-1} p_i \cos\omega t - a\left(x_i\cos\omega t - \omega^{-1} p_i \sin\omega t\right)\right].$$

The identity transformation corresponds to a = 0.

$$V'_{N+1,i}: \quad t' = t$$

$$x_j' = x_j + a\,\delta_{ij}\cos\omega t \qquad (6.19g)$$

$$p_j' = p_j - a\omega\,\delta_{ij}\sin\omega t$$

$$V'_{i,N+1}: \quad t' = \omega^{-1}\tan^{-1}\left(\frac{\tan\omega t}{1+ax_i\sec\omega t}\right)$$

$$x_j' = \frac{x_j\sec\omega t}{\sqrt{(1+ax_i\sec\omega t)^2+\tan^2\omega t}} \qquad (6.19h)$$

$$p_j' = \frac{-\omega x_j(\tan\omega t\sec\omega t)[1+ax_i\sec\omega t]^{-1}}{\sqrt{(1+ax_i\sec\omega t)^2+\tan^2\omega t}} +$$

$$+\omega\sqrt{1+\left(\frac{\tan\omega t}{1+ax_i\sec\omega t}\right)^2}\left[\frac{x_j\sin\omega t+\omega^{-1}p_j\cos\omega t+a\omega^{-1}(x_ip_j-x_jp_i)}{1+a(x_i\cos\omega t-\omega^{-1}p_i\sin\omega t)}\right].$$

Note we get the identity transformation for a = 0.

Both of these sets of formulas are rather remarkable in that they reduce by inspection to the identity transformation by the proper choice of the value of the parameter a, yet these transformations would be very difficult to derive by integrating the corresponding Lie differential equations.

We note that all of these finite transformations are system covariance transformations (see appendix I). Thus we could use them as we did the solution transformation or the Bluman transformation to derive new realizations of a given Lie algebra. For example, we could use the transformations of (4.6) to transform the generators (6.11), as well as using the finite transformations corresponding to the generators (6.11) to transform the set (4.4). We thus see that there is the possibility of deriving a very large number of realizations of a given Lie algebra in terms of the dynamical variables of a given physical system.

CHAPTER VII

CONCLUSIONS

The previous chapter showed that by using a transformation of the free particle variables, one may find a realization of the transformation group SL(N+2,R) in terms of harmonic oscillator variables. This procedure is applicable to systems with the same number of degrees of freedom.

By knowing two distinct locally one-to-one transformations between two Hamiltonian systems, one may derive a set of "ladders" of distinct realizations of a given Lie algebra; two "rungs" of two of these ladders for the free particle were given as U, U' and U, U'' in the previous chapter.

Thus, when two physical systems with the same number of degrees of freedom can be connected by a locally one-to-one transformation, we can say something about the covariance transformation group of one system's Hamiltonian equations of motion if we know something about the covariance transformation group of the other system's Hamiltonian equations of motion. As discussed in reference [4], when the Hamiltonian systems are completely integrable, then these considerations become global in nature.

In this sense, we can say that many Hamiltonian systems with the same number of degrees of freedom are essentially equivalent. In attacking a new problem, we may then assume the existence of an SL(N+2,R) covariance transformation group for the Hamiltonian equations of motion and perhaps derive useful information from this knowledge.

The existence of a large number of realizations of a given algebra or group in terms of a particular system's dynamical variables forces us to give criteria to select one realization over another. One important criterion is that the transformation group elements act transitively on the solution space of the physical system.

Reference [4] makes an important point: the use presented there and here of the solution transformation is an algoritmic technique. If one can write down the solutions to an initial value Hamiltonian problem, finding a connection to the free particle problem with the same number of degrees of freedom becomes straightforward. The calculation of the SL(N+2,R) transformation group elements also becomes straightforward.

Chapter V presented a connection between the constants of the motion of a given Hamiltonian system and the Lie symmetry generators of that system. This connection could prove calculationally useful: for an unsolved system of equations of motion, one may be able to obtain a solution to the Lie determining equations (1.6) where ξ is chosen to be zero. If the solution obtained obeys the integrability conditions (5.3), then one can find an associated constant of the motion. As is well-known, the use of this constant of the motion can reduce the complexity of the equations of motion.

APPENDIX I

A COMMENT ON THE EXPRESSION "CANONICAL TRANSFORMATION"

In studying the literature on classical Hamiltonian dynamics, one often comes across the word "transformation" described by the adjective "canonical"; in addition, one often sees conflicting uses of this term. Thus, throughout this work we have been careful not to use this adjective to describe the word transformation.

A canonical transformation is usually defined in the literature to be a transformation under which the Hamiltonian equations of motion of <u>all</u> Hamiltonian systems with the same number of degrees of freedom are covariant (invariant in form) in that a function $K(Q,P,t)$ exists such that under the canonical transformation the equations of motion $\dot{q}_i = H_{p_i}$, $\dot{p}_i = -H_{q_i}$ are transformed into the equations of motion $\dot{Q}_i = K_{P_i}$, $\dot{P}_i = -K_{Q_i}$ <u>regardless of the explicit algebraic form of the Hamiltonian $H(q,p,t)$</u> (in this appendix, we revert to the standard q,p notation of mechanics). These transformations certainly add to the elegant theoretical structure of classical Hamiltonian dynamics, but they do not allow one to develop a systematic algorithmic method for solving all

Hamiltonian dynamical problems. This statement is actually obvious, as the following example shows.

The canonical transformation $q = (2P)^{1/2}\sin Q$, $p = (2P)^{1/2}\cos Q$ whose generating function* is

$$F_2(q,P,t) = \tfrac{1}{2}q\sqrt{2P-q^2} + P\sin^{-1}\left(\frac{q}{\sqrt{2P}}\right)$$

changes the harmonic oscillator equations of motion $\dot{q} = p$, $\dot{p} = -q$ into the equivalent Hamiltonian equations of motion $\dot{Q} = 1$, $\dot{P} = 0$, which are trivial to integrate. On the other hand, the free particle equations of motion $\dot{q} = p$, $\dot{p} = 0$ are transformed under this transformation into the equivalent Hamiltonian equations of motion $\dot{Q} = \cos^2 Q$, $\dot{P} = P\sin 2Q$; these equations are much more difficult to integrate than are the original free particle equations of motion.

One may instead wish to study transformations which preserve the Hamiltonian form of the equations of motion of a certain given Hamiltonian system; in the language of Saletan and Cromer*, we call these transformations canonoid transformations. The set of all canonical transformations for

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* E. J. Saletan and A. H. Cromer, Theoretical Mechanics, John Wiley and Sons, 1971

Hamiltonian systems with a particular number of degrees of freedom is evidently a subset of the set of all canonoid transformations for any given Hamiltonian system with the same number of degrees of freedom.

Another type of transformation used in the study of the physical and the mathematical symmetries of a given Hamiltonian system are those transformations which transform the given Hamiltonian system into another Hamiltonian system with a Hamiltonian function that is identical in form with the old Hamiltonian function; in other words, the Hamiltonian $H(q,p,t)$ becomes the Hamiltonian $H(Q,P,T)$. We will call these transformations <u>system covariance transformations</u>. If these transformations can be continuously connected to the identity transformation, then they can be studied by using the elegant machinery of Lie. It is mainly system covariance transformations which are of interest in the present work, although the results given would not have been found without the use of canonoid transformations.

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